3.2 PROPERTIES OF DETERMINANTS.

We want to know if

a) $\det(A+B) \overset{?}{=} \det(A) + \det(B)$

b) $\det(AB) \overset{?}{=} \det(A)\det(B)$

c) $\det(kA) \overset{?}{=} k\det(A)$

for A, B $n \times n$ matrices and k scalar.

Before studying general results, let's consider some particular cases:

1) $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 5 & -6 \end{bmatrix}$

$\Rightarrow \quad A+B = \begin{bmatrix} 3 & 0 \\ 7 & -2 \end{bmatrix}$

and $\det(A) = -2$, $\det(B) = 3$, $\det(A+B) = -6$

Therefore $\det(A+B) = -6 \neq \det(A) + \det(B) = -2+3 = 1$

2) However, for $A_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$, and

$C = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}$, we have

$\det(A_2) = -2$, $\det(B_2) = 8$, $\det(C) = 6$

Thus $\det(C) = \det(A_2) + \det(B_2)$.

3) Also,

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 17 & -21 \\ 24 & -30 \end{bmatrix}$$

$\det(A) = -2$, $\det(B) = 3$, $\det(AB) = -510 + 504 = -6$

Therefore,

$$\det(AB) = -6 = (-2) * (3) = \det(A)\det(B).$$

4) Consider

$$3A = \begin{bmatrix} 3 & 9 \\ 6 & 12 \end{bmatrix}$$

Then

$$\det(3A) = 36 - 54 = -18 \neq 3 * (-2) = \\ = 3\det(A)$$

In fact,

$$\det(3A) = -18 = 9 * (-2) = 3^2 \det(A).$$

5) If

$$D = \begin{bmatrix} 5*1 & 5*3 \\ 2 & 4 \end{bmatrix}$$

$$\Rightarrow \det(D) = 5(1*4) - 5(3*2) = 5 * (4-6) = \\ = 5 * (-2) = 5\det(A).$$

Does it mean $\det \begin{pmatrix} ka_{11} & ka_{12} & .. & ka_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & .. & a_{nn} \end{pmatrix} = K\det(A)$?

In this section, we also want to generalize thm 2 of section 2.2, it means

$$\boxed{A_{2\times 2} \text{ is invertible if and only if } \det A \neq 0,}$$

to general $n\times n$ matrices.

In order to do this, we need to learn how to compute $\det A$ from the determinants of the elementary matrices used in the row reduction of matrix A to an echelon form.

Relationship between det A and det B, where B is obtained from A by row reduction.

Exercise

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix} \xrightarrow{\text{3 replacements}} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{1 replac.}} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{bmatrix} \xrightarrow{\text{1 replac.}} \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{bmatrix} = B$$

How are det(A) and det(B) related?

$$\det(B) = 1 \cdot 1 \cdot (-3) \cdot (-13) = 39 \quad \text{thm 2}$$

$$\det(A) = \det(B) \quad \text{thm 3 (Next page)}$$

Observation: In this particular example, the only row operations are "replacements". If row interchanges or scaling were performed then it would not be longer true that

det A = det B in general.

Thm 3

A is an $n \times n$ matrix

a) If B is the matrix that results when a row or a column is multiplied by k.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

then, $\det(B) = k \det(A)$.

b) If B is obtained from A by interchanging two rows or two columns, then $\det(B) = -\det(A)$

c) If B results by adding a multiple of one row to another row of A (Similar for columns), then

$$\det(B) = \det(A).$$

(look at book's table for 3×3 matrices).

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Anton : Linear Algebra

EXAMPLE 2 Theorem 3 Applied to 3 × 3 Determinants

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	The first row of A is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	The first and second rows of A are interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	A multiple of the second row of A is added to the first row.

We will verify the equation in the last row of the table and leave the first two for the reader. With the help of Example 7 in Section 2.1 we obtain

$$\begin{aligned}
\det(B) &= (a_{11} + ka_{21})a_{22}a_{33} + (a_{12} + ka_{22})a_{23}a_{31} + (a_{13} + ka_{23})a_{21}a_{32} \\
&\qquad - a_{31}a_{22}(a_{13} + ka_{23}) - a_{33}a_{21}(a_{12} + ka_{22}) - a_{32}a_{23}(a_{11} + ka_{21}) \\
&= \det(A) + k(a_{21}a_{22}a_{33} + a_{22}a_{23}a_{31} + a_{23}a_{21}a_{32} \\
&\qquad - a_{31}a_{22}a_{23} - a_{33}a_{21}a_{22} - a_{32}a_{23}a_{21}) \\
&= \det(A) + 0 = \det(A)
\end{aligned}$$

♦

REMARK As illustrated by the first equation in Example 2, part (*a*) of Theorem 2.2.3 allows us to bring a "common factor" from any row (or column) through the determinant sign.

Elementary Matrices

Recall that an elementary matrix results from performing a single elementary row operation on an identity matrix; thus, if we let $A = I_n$ in Theorem 2.2.3 [so that we have $\det(A) = \det(I_n) = 1$], then the matrix B is an elementary matrix, and the theorem yields the following result about determinants of elementary matrices.

Theorem

Let E be an $n \times n$ elementary matrix.

(*a*) *If E results from multiplying a row of I_n by k, then* $\det(E) = k$.
(*b*) *If E results from interchanging two rows of I_n, then* $\det(E) = -1$.
(*c*) *If E results from adding a multiple of one row of I_n to another, then* $\det(E) = 1$.

Proof of thm 3 part (a)

Let
$$B = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Applying Cofactor expansion along first row to Compute $\det B$ leads to

$$\begin{aligned} \det B &= ka_{11}C_{11} + ka_{12}C_{12} + \cdots + ka_{1n}C_{1n} \\ &= k\left(a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}\right) \\ &= k \det A. \end{aligned}$$

Proof of Theorem 3 part (b) for a 3x3 matrix A.

For a 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove the following: $\quad = B \quad$ Row interchange

$$\det A = -\det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -\det B$$

Proof:- Renaming the entries of B

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where
$b_{11} = a_{21}$, $b_{12} = a_{22}$, $b_{13} = a_{23}$
$b_{21} = a_{11}$, $b_{22} = a_{12}$, $b_{23} = a_{13}$
$b_{31} = a_{31}$, $b_{32} = a_{32}$, $b_{33} = a_{33}$

then

$$\det B = b_{11} b_{22} b_{33} + b_{21} b_{32} b_{13} + b_{12} b_{23} b_{31} - (b_{13} b_{22} b_{31} + b_{12} b_{21} b_{33} + b_{23} b_{32} b_{11})$$

$$= a_{21} a_{12} a_{33} + a_{11} a_{32} a_{23} + a_{22} a_{13} a_{31} - (a_{23} a_{12} a_{31} + a_{22} a_{11} a_{33} + a_{13} a_{32} a_{21}) =$$

$$= -\Big(a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{12} a_{23} a_{31} - (a_{13} a_{22} a_{31} + a_{12} a_{21} a_{33} + a_{23} a_{32} a_{11}) \Big)$$

$$= -\det A.$$

Exercises:

1) $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$, $\det A = ?$

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = (-3)(-55) = 165.$$

2) $$\begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = -1 \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = (-1)(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = (+1)(-18) = \underline{\underline{-18}}$$

<u>Lemma I</u>:- If $B_{n\times n}$ matrix is such that two of its rows are identical then

$$\det B = 0.$$

<u>Proof</u>:- It is done for a 3×3 matrix. For the general case similar ideas can be applied.

let

$$B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then using cofactor expansion along first row

$$\det B = a_{21}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{21}a_{22}a_{33} - a_{21}a_{23}a_{32} - a_{22}a_{21}a_{33} + a_{22}a_{23}a_{31} + a_{23}a_{21}a_{32} - a_{23}a_{22}a_{31}$$

$$= 0$$

Thus, $\boxed{\det B = 0}$

Lemma 2.- Let

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \vdots & \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\det \underbrace{\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}}_{=D} = \det A + \det C.$$

Proof.- Applying Cofactor expansion along the first row of D to compute its determinant leads to

$$\begin{aligned} \det D &= (a_{11}+b_{11})C_{11} + (a_{12}+b_{12})C_{12} + \cdots + (a_{1n}+b_{1n})C_{1n} \\ &= (a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}) + (b_{11}C_{11} + b_{12}C_{12} + \cdots + b_{1n}C_{1n}) \\ &= \det A + \det C \end{aligned}$$

Proof of theorem 3 part (c).

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} a_{11} + Ka_{21} & \cdots & a_{1n} + Ka_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

then, using lemma 2

$$\det B = \det A + \det \begin{bmatrix} Ka_{21} & \cdots & Ka_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}$$

$$\overset{\text{part (a) Thm 3}}{=\!=} \det A + K \det \begin{bmatrix} a_{21} & \cdots & a_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \overset{\text{lemma 1}}{=\!=} \det A + K \times 0$$

$$= \det A.$$

Therefore, $\det A$ does not change if a multiple of a row is added to another row.

Assume A has been reduced to an echelon form U by row replacements and row interchanges

Then, from thm 3

$$\det A = (-1)^r \det U$$

Since U is in echelon form and $U_{n\times n}$

$$\det U = u_{11} \cdots u_{nn} \quad (\text{Triangular matrix})$$

Then

$$\det A = \begin{cases} (-1)^r \left(\begin{array}{c}\text{product of pivots} \\ \text{in } U\end{array}\right), & A \text{ invertible} \\ 0, & A \text{ is not invertible} \end{cases}$$

Now, we want to establish a <u>relationship between invertibility of matrices and their determinants.</u>

First, we will learn how to compute $\det A$ from determinants of elementary matrices.

<u>Corollary 1 from thm 3.-</u>

Let E be an $n \times n$ elementary matrix

a) If E results from multiplying a row of I_n by k, then $\det E = k$.

b) If E results from interchanging two rows of I_n, then $\det E = -1$.

c) If E results from adding a multiple of one row of I_n to another row, then

$$\det E = 1.$$

<u>Proof.-</u> It is an immediate consequence of thm 3 and the fact that

$$\det I_n = 1.$$

Corollary 2 from thm 3

If $B_{n\times n}$ matrix and $E_{n\times n}$ elementary matrix, then

$$\det EB = \det E \det B.$$

Proof.- There are three cases depending on E.

Case 1.-

$$E = \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & k & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix} i^{th}\text{ row}, \quad \det E \overset{\text{Corollary 1 (a)}}{=\!=} k$$

then

$$EB = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ kb_{i1} & kb_{i2} & \cdots & kb_{in} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

and

$$\det EB \overset{\text{thm 3}}{=\!=} k \det B = \det E \det B$$

Case 2.-

If E is obtained from interchanging two rows of I_n

then $\det E \overset{\text{Corollary 1 (b)}}{=} -1$ and

EB is the matrix that results from interchanging the same rows of B.

Therefore, $\det EB \overset{\text{Thm 3}}{=} -\det B = \det E \det B$.

Case 3.-

If E is obtained by adding a multiple of one row of I_n to another row then

$$\det E \overset{\text{Corollary 1 (c)}}{=} 1$$

and EB is the matrix that results from adding a multiple of the same row ~~of~~ B to the same another row of B. Therefore,

$$\det EB \overset{\text{thm 3}}{=\!=} \det B = \det E \det B .$$

Corollary 3.- If $B_{n\times n}$ matrix and $E_1, E_2, ., E_r$ are $n\times n$ elementary matrices,

then

$$\det(E_1 E_2 E_3 .. E_r B) = \det E_1 \det E_2 \ldots \det E_r \det B .$$

Proof.-

$$\det(E_1 E_2 . E_r B) \overset{\text{Cor. 2}}{=} \det E_1 \det(E_2 E_3 \ldots E_r B)$$

$$\det E_1 \det E_2 \det(E_3 \ldots E_r B) = \ldots \overset{\text{Cor 2.}}{=} \det E_1 \det E_2 .. \det E_r \det B$$

Thm 4 $\quad A_{n\times n}$ is invertible if and only if $\det A \neq 0$.

Proof.-

$(\rightarrow)$ A invertible implies that

$$A = E_1 E_2 \ldots E_r I_n \quad (\text{Inv. matrix thm}).$$

then using Corollary 3

$$\det A = \overset{\neq 0}{\det E_1}\ \overset{\neq 0}{\det E_2} \ldots \overset{\neq 0}{\det E_r}\ \overset{=1}{\det I_n} \neq 0$$

$(\leftarrow)$ Consider R the row-reduced echelon form of A

then $R = I_n$ or R has a row of zeros (Key lemma)

Also $\quad R = E_1 E_2 \ldots E_r A$

Therefore, $\det R \overset{\text{Cor. 3}}{=} \overset{\neq 0}{\det E_1}\ \overset{\neq 0}{\det E_2} \ldots \overset{\neq 0}{\det E_r}\ \overset{\neq 0}{\det A} \neq 0$

then, R cannot have a row of zeros, otherwise

$$\det R = 0.$$

Therefore, the only possibility for R is to be I_n

i.e., $\quad R = I_n$.

and that means $A \sim R = I$ is invertible

according to Inv. matrix Thm.

Thm 6.- $A_{n\times n}$ and $B_{n\times n}$

$$\det AB = \det A \det B$$

Proof.- Two cases arise depending on A.

CASE 1.- A is singular $\Longrightarrow$ AB is singular. (Statement in pag 6 Sect. 2.3)

$$\Rightarrow \det AB = 0 = 0.\det B = \det A \det B \checkmark$$

CASE 2.- A is invertible $\overset{IMT}{\Longrightarrow}$ $A \sim I_n$

$\Rightarrow$ $E_r \dots E_1 A = I_n$ $\overset{\text{Using inverse of elem. matrices}}{\Rightarrow}$ $A = E_1^{-1} \dots E_r^{-1}$ also elem. matrices.

$$\Rightarrow \det AB = \det(E_1^{-1} \dots E_r^{-1} B) \overset{\text{Corollary 2}}{=} \det E_1^{-1} \det(E_2^{-1} \dots E_r^{-1} B) =$$

$$\overset{\text{after } r-1 \text{ steps}}{=} \det E_1^{-1} \det E_2^{-1} \dots \det E_r^{-1} \det B =$$

$$\overset{\text{Corollary 3}}{=} \det(E_1^{-1} E_2^{-1} \dots E_r^{-1}) \det B = \det A \det B$$

We can easily prove that

$$\det(A) = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23} \quad \left(\begin{array}{l}\text{elimination}\\ \text{along } 2^{nd} \text{ row}\end{array}\right)$$

$$\det(A) = a_{31}M_{31} \quad M_{32} + a_{33}M_{33} \quad (\text{along } 3^{rd} \text{ row})$$

$$\det(A) = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32} \quad (\text{along } 2^{nd} \text{ col.})$$

$$\det(A) = a_{13}M_{13} - a_{23}M_{23} + a_{33}M_{33} \quad (\text{along } 3^{rd} \text{ col.})$$

However,

$$a_{11}M_{31} - a_{12}M_{32} + a_{13}M_{33} = \quad \left(\begin{array}{l}\text{entries and } M\text{'s}\\ \text{correspond to different}\\ \text{rows}\end{array}\right)$$

$$= 1(3) - 4(0) - 3(1) = 0$$

This is a particular case of a general result.

<u>Definition</u>:- The determinant M_{ij} of the submatrix that remains after the ith row and jth column are deleted from A is called <u>the minor of entry a_{ij}</u>

The number $C_{ij} = (-1)^{i+j} M_{ij}$ is called <u>the cofactor of entry a_{ij}</u>.

In our previous example, we showed a different way to obtain the det (A) for a 3×3 matrix. This result can be extended to any matrix $n \times n$.

Thm 2.4.1 Expansion by cofactors.

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$ (Cofactor Expansion along j^{th} column)

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$ (Cofactor exp. along ith row)

$$1 \le i \le n, \quad 1 \le j \le n.$$

Efficient method to evaluate determinants:

Cofactor expansion + row or column operations.

Exercise #9. Find det (A)

$$\det(A) = \begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 6 & 11 & 0 & 2 \\ 2 & 10 & 3 & 2 \end{vmatrix} = -3 \begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 6 & 11 & 2 \end{vmatrix} =$$

$$= -3 \begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 8 & 13 & 0 \end{vmatrix} = -3\left[5\begin{vmatrix} 2 & 2 \\ 8 & 13 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 8 & 13 \end{vmatrix}\right] =$$

$$= -3\left(5(26-16) + 2(39-24)\right) = -3(50+30)$$

$$= \underline{\underline{-240}}$$